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WAVE FORMATION ON VERTICAL FALLING LIQUID FILMS

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Abstract—The method of integral relations is used to derive a nonlinear "two-wave" structure equation for long waves on the surface of vertical falling liquid films. This equation is valid in a wide range of Reynolds numbers and reduces to the known equations for high and low Re. Theoretical data for the fastest growing waves are compared with the experimental results on velocities, wave numbers and growth rates of the waves in the inception region. The validity of theoretical assumptions is also confirmed by the direct measurements of the instantaneous velocity profiles in a wave liquid film.

1. INTRODUCTION

This work is concerned with studying the long waves always existing on the surface of vertical falling liquid films, which are extensively used in interphase heat and mass transfer processes in chemical technology and energetics. These waves can significantly influence the transfer processes, and in the case of short test sections and moderate liquid flow rates they are ordered and two-dimensional. To estimate the transfer coefficients for the wave flows of films one should be able to calculate the nonlinear wave regimes dominating in wave formation on the surface of the above films.

Most known publications on the nonlinear waves are devoted to the calculations of either nonstationary or stationary waves at low and moderate flow rates, respectively. But there is no universal nonlinear nonstationary equation which would generalize the available approaches and be valid in a wide range of conditions. The aim of the present study was to derive a universal model equation for nonlinear nonstationary waves on the film surface and to substantiate the validity of the applied approach on the basis of experimental and theoretical data.

The stability of vertical falling liquid films was studied in terms of the Orr-Sommerfeld equation by Benjamin (1957), Yih (1963), Whitaker (1964), and Krantz & Goren (1971). Kapitza (1948), Shkadov (1967, 1968), Krylov *et al.* (1969) and Lee (1969) performed such an analysis in terms of the boundary layer equations.

According to the theory of stability the vertical falling film is unstable at any Reynolds number. In experiments the finite-amplitude waves are observed which may be analyzed only in terms of the nonlinear equations. It is very convenient to perform this analysis using one equation, e.g. for the film thickness.

Nonlinear wave studies are conditionally subdivided into two groups for moderate and low Reynolds numbers. Kapitza (1948) was the first to analyze the film surface stationary waves at moderate Re. He derived a third order stationary equation for the film thickness and studied its linear approximation. Actually he found the condition for the neutral wave regimes and obtained the expression for neutral wave velocity and length on a film. He also tried to predict the equilibrium wave amplitude from the balance between the energy dissipation rate and the gravitational work and the additional hypothesis on the minimum dissipation function for the actual wave flow regime. Kapitza's theory is not rigorous in many respects, it is based on physical arguments rather than on a strict derivation (e.g. his attempts to define the critical Reynolds number and the wave amplitude). Nevertheless his approach, based on the boundary layer type equations and the method of integral relations, has given rise to numerous studies on its development.

From Kapitza's followers, Shkadov (1967, 1968, 1973) and Lee (1969) should be mentioned. Shkadov studied nonlinear stationary waves at moderate Reynolds numbers on the basis of the boundary layer type equations for long waves and the method of integral relations. Periodic solutions were found as expansions in harmonics. Retaining the first two harmonics in the solution, Shkadov determined the basic wave characteristics including amplitude, by the additional harmonic amplitude equation and the minimum film thickness hypothesis similar to that suggested by Kapitza.

A similar study was performed by Lee (1969) where the problem of the stationary waves was solved on the basis of Kapitza's equations by the Bogolyubov-Krylov method.

Thus all the nonlinear wave regime studies at moderate and high Re were performed on the basis of Kapitza's type equations in the boundary layer approximation. The dynamic and continuity equations are reduced to one, assuming that the waves are stationary and the solution is a simple wave.

Beginning with Ivanilov's study (1961), the "narrow bands" method which consists in expansion of the solution in h/λ powers, where h is the film thickness and λ is the wave length, is used for the analysis of long waves at low Reynolds numbers. The equations comprise the $h/\lambda \cdot \mathrm{Re}$ products, therefore to ensure the expansion convergence it is necessary that Re $\sim$ I.

Some long-wave theory equations of various accuracy were obtained from the complete system of the Navier-Stokes equations by Ivanilov (1961), Benney (1966), Pashinina (1966), Gjevik (1970) and Maurin *et al.* (1977). A detailed analysis of the solutions and their stability is given by Nepomnyashchii (1974, 1977). Petviashvili and Tzvelodub (1978) and Tzvelodub (1980) obtained solutions for both the stationary periodic waves and the stationary two- and three-dimensional solutions on a liquid film at Re $\sim$ I.

Thus the present situation in the theory is as follows. The flow stability has been studied on the basis of the Orr–Sommerfeld and boundary layer equations. The nonlinear wave motion at low Reynolds numbers has been analyzed by the Benney–Gjevik type equation which permits us to investigate both the nonstationary and the stationary wave solutions.

For the moderate Re no model nonstationary equation is known. There are only those of the Kapitza's type equation for the analysis of stationary wave regimes. One attempt to derive a model equation for high Re was made by Nakoryakov & Shreiber (1973). There is a certain necessity for a general nonstationary nonlinear wave equation which would generalize the known approaches. The requirements which should be met by this equation are obvious. It should ensure the analysis of stability and reduce into the Benney–Gjevik equation at low Re. In the stationary case it should coincide with Kapitza's type equation and at high Re reduce into the Nakoryakov & Shreiber's equation (1973). An attempt to derive such an equation is reported in the present paper.

As far as the experimental studies are concerned, only a few publications whose results may be compared with the theoretical models should be mentioned. They are Kapitza & Kapitza (1949), Jones & Whitaker (1966), Strobel & Whitaker (1969), Krantz & Goren (1971), Portalski & Clegg (1972), Pierson & Whitaker (1977) and Nakoryakov *et al.* (1975, 1976, 1977). This is due to the fact that the film wave theory has been fairly well developed only for the two-dimensional periodic waves of small amplitude of almost sinusoidal form. In practice the two-dimensional regular wave regimes are usually observed at Re $\sim 5 - 20$ near the wave inception line. In other cases the film waves are three-dimensional, irregular and their form is far from being sinusoidal. Therefore almost all experimental studies are actually the statistical analyses of wave characteristics without any separation of two- and three-dimensional, stationary and nonstationary waves. Certainly, this information cannot ensure complete verification of the theoretical models and cannot give a physical description of the wave motion.

In the above experimental studies it was possible to investigate the two-dimensional regular wave regimes either near the wave inception line in a limited range of variations of the flow rates, liquid properties and wave characteristics or using an artificial regularization method, i.e. the superposition of external disturbances. To verify the stability theories, Krantz & Goren (1971) excited waves by wire vibrations at the initial film section. In our studies similar to Kapitza, Kapitza (1949) the stationary wave regimes were mainly studied via wave excitation by pulsations of the liquid flow rate.

Wave studies in the inception region show that the behaviour of natural growing waves is described by the linear theories of the fastest growing waves. Most studies, however, provide data only for the velocity and wave number, without mentioning that it is the two-dimensional flow that is under study. The theories of steady finite amplitude waves are in agreement with the experimental results only for the almost sinusoidal waves in the range of Re ~ 5–30.

The literature lacks data on the experimental instantaneous velocity profiles measurements in a wave liquid film, though such measurements are of extreme importance, since many theories utilize various velocity distribution hypotheses.

Here we present experimental data on the wave characteristics and instantaneous velocity profiles for the strictly two-dimensional waves.

2. DERIVATION OF WAVE EQUATION

Let the Navier–Stokes equations and the boundary conditions for a verticle falling liquid film (figure 1) be written in the dimensionless form:

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{3}{\epsilon \cdot \mathrm{Re}} + \frac{1}{\epsilon \cdot \mathrm{Re}}\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\epsilon^2 + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right) - \frac{\partial \bar{p}}{\partial \bar{x}}, \quad [1]$$

$$\epsilon^2\left(\frac{\partial \bar{v}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}\right) = \frac{\epsilon}{\mathrm{Re}}\frac{\partial^2 \bar{v}}{\partial \bar{y}^2} + \frac{\epsilon^3}{\mathrm{Re}}\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} - \frac{\partial \bar{p}}{\partial \bar{y}}, \quad [2]$$

$$\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \quad [3]$$

at $y = h$,

$$\epsilon^2\frac{4\partial \bar{h}/\partial \bar{x}}{1 - \epsilon^2(\partial \bar{h}/\partial \bar{x})^2} \cdot \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{y}} + \epsilon^2\frac{\partial \bar{v}}{\partial \bar{x}} = 0, \quad [4]$$

$$\Delta \bar{P} = -\frac{3^{1/3}\,\mathrm{Fi}^{1/3}\,\epsilon^2}{\mathrm{Re}^{5/3}}\frac{\partial^2 \bar{h}/\partial \bar{x}^2}{[1 + \epsilon^2(\partial \bar{h}/\partial \bar{x})^2]^{3/2}} + \frac{2\epsilon}{\mathrm{Re}}\frac{\partial \bar{v}}{\partial \bar{y}}\left[\frac{1 + \epsilon^2(\partial \bar{h}/\partial \bar{x})^2}{1 - \epsilon^2(\partial \bar{h}/\partial \bar{x})^2}\right], \quad [5]$$

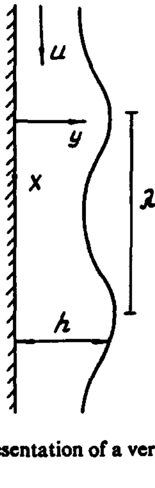

Figure 1. Schematic representation of a vertical falling liquid film.

at $y = 0$,

$$\bar{u} = 0, \quad \bar{v} = 0, \tag{6}$$

at $y = h$,

$$\bar{v} = \frac{\partial \bar{h}}{\partial \bar{t}} + \bar{U}\frac{\partial \bar{h}}{\partial \bar{x}}. \tag{7}$$

The conditions are: [4] is the vanishing tangential stress component, [5] is the continuity in the normal stress component and [7] is the kinematic boundary condition on a free surface. Here

$$\bar{u} = u/u_0, \quad \bar{v} = (v/u_0)L/h_0, \quad \bar{x} = x/L, \quad \bar{y} = y/h_0,$$
$$\bar{t} = tu_0/L, \quad \bar{P} = P/(\rho u_0^2), \quad \epsilon = h_0/L, \quad \mathrm{Re} = q_0/\nu, \quad \bar{h} = h/h_0,$$

film number $\mathrm{Fi} = \sigma^3/(\rho^3 g \nu^4)$ are introduced, h_0 and u_0 are determined from the Nusselt formulas for a smooth laminar film flow

$$\mathrm{Re} = q_0/\nu = gh_0^3/(3\nu^2) = h_0 u_0/\nu.$$

L is the characteristic longitudinal scale, e.g. the wavelength λ. Let a long-wave process be considered, provided that $L \sim \lambda$, $\epsilon \ll 1$ and $\mathrm{Re} \sim 1/\epsilon \gg 1$.

Separate the disturbed part u' of the longitudinal velocity u and set $u' \sim \epsilon_1 u_0$, $\epsilon_1 \sim \epsilon$. In [1–7] retain the terms of orders 1 and ϵ. Hence we come to the boundary-layer type equations:

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{3}{\epsilon \cdot \mathrm{Re}} + \frac{1}{\epsilon \cdot \mathrm{Re}}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \frac{\partial \bar{P}}{\partial \bar{x}}, \tag{8}$$

$$\frac{\partial \bar{P}}{\partial \bar{y}} = 0, \tag{9}$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{10}$$

with the boundary conditions

$$\frac{\partial \bar{u}}{\partial \bar{y}} = 0 \qquad \text{at } y = h, \tag{11}$$

$$\Delta\bar{P} = -\frac{3^{1/3}\mathrm{Fi}^{1/3}\epsilon^2}{\mathrm{Re}^{5/3}}\frac{\partial^2 \bar{h}}{\partial \bar{x}^2} \qquad \text{at } y = h. \tag{12}$$

Conditions [6] and [7] remain unchanged. Using [9] and [12], rewrite [8] in the form:

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{3}{\epsilon \cdot \mathrm{Re}} + \frac{1}{\epsilon \cdot \mathrm{Re}}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{(3\,\mathrm{Fi})^{1/3}}{\mathrm{Re}^{5/3}}\epsilon^2\frac{\partial^3 \bar{h}}{\partial \bar{x}^3}. \tag{13}$$

For the derivation of [13] it was taken into consideration that for real liquids $\mathrm{Fi}^{1/3}$ is high (e.g. for water $\mathrm{Fi}^{1/3} \approx 10^4$).

It should be noted that with the properly chosen relation between ϵ and ϵ_1 the final system of equations [13], [11], [10], [6] and [7] remains the same in the range of Reynolds numbers

$\mathrm{Re} = 1 - 1/\epsilon^2$. But the more rigorous range of applicability of the above system should be specified in the coordinates $\{\mathrm{Re}, \mathrm{Fi}, \epsilon, \epsilon_l\}$.

In what follows the method of integral relations (the Karman–Pohlhausen method) is used. Its main disadvantage is the necessity to set the film instantaneous velocity profile a priory. It is difficult to estimate its possible error, though it is likely that for the long waves it cannot be too high. The experimental results by Nakoryakov *et al.* (1977) on the direct determination of instantaneous velocity profiles in a wave liquid film (partially given in the present paper) show a fair approximation to the velocity profile by a self-similar polynomial for the two-dimensional waves of at least moderate amplitude.

Now we turn from the initial system of equations [13], [11], [10], [6] and [7] to the integral relations. In this case it will be convenient to rewrite these equations in the dimensional form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g + \frac{\sigma}{\rho}\frac{\partial^3 h}{\partial x^3}, \qquad [14]$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad [15]$$

$$v = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} \qquad \text{at } y = h,$$

$$\frac{\partial u}{\partial y} = 0 \qquad \text{at } y = h,$$

$$u = v = 0 \qquad \text{at } y = 0.$$

Integrate [14] and [15] over the film thickness

$$\int_0^h \frac{\partial u}{\partial t}\,\mathrm{d}y + \frac{1}{2}\int_0^h \frac{\partial u^2}{\partial x}\,\mathrm{d}y + \int_0^h v\frac{\partial u}{\partial y}\,\mathrm{d}y = -\nu\left(\frac{\partial u}{\partial y}\right)_{y=0} + gh + \frac{\sigma h}{\rho}\frac{\partial^3 h}{\partial x^3}, \int_0^h \frac{\partial u}{\partial x}\,\mathrm{d}y + v\Big|_{y=h} = 0. \qquad [16]$$

In the first integral of [16] take the derivative out of the integral sign

$$\int_0^h \frac{\partial u}{\partial t}\,\mathrm{d}y = \frac{\partial}{\partial t}\int_0^h u\,\mathrm{d}y - U\frac{\partial h}{\partial t}.$$

Perform the similar transformations in other integrals of [16], then allowing for the boundary conditions and taking the third integral in [16] by parts we obtain the equations:

$$\frac{\partial}{\partial t}\int_0^h u\,\mathrm{d}y + \frac{\partial}{\partial x}\int_0^h u^2\,\mathrm{d}y = -\nu\left(\frac{\partial u}{\partial y}\right)_{y=0} + gh + \frac{\sigma h}{\rho}\frac{\partial^3 h}{\partial x^3}, \qquad [17]$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^h u\,\mathrm{d}y = 0. \qquad [18]$$

Proceeding from the method of integral relations and the results of our experiments, the velocity profile is

$$u = U\cdot f(\eta) \qquad \eta = y/h$$

The function $f(\eta)$ can be approximated, e.g. by the second power polynominal with coefficients satisfying boundary conditions [6] and [11]

$$f(\eta) = 2\eta - \eta^2 \tag{19}$$

Now we introduce the instantaneous liquid flow rate in a film and express it as a function of f:

$$q = \int_0^h u\,\mathrm{d}y = uh \int_0^1 f\,\mathrm{d}\eta.$$

In a similar way we have:

$$\int_0^h u^2\,\mathrm{d}y = U^2 h \int_0^1 f^2\,\mathrm{d}\eta, \left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{U}{h}\frac{\mathrm{d}f}{\mathrm{d}\eta}\bigg|_{\eta=0}.$$

Using these expressions and introducing the coefficients

$$\int_0^1 f\,\mathrm{d}\eta = \delta, \quad \int_0^1 f^2\,\mathrm{d}\eta = \gamma, \quad \frac{\mathrm{d}f}{\mathrm{d}\eta}\bigg|_{\eta=0} = \kappa, \quad \chi = \gamma/\delta^2,$$

rewrite [17] and [18] as thickness and flow rate equations

$$\frac{\partial q}{\partial t} + 2\chi\frac{q}{h}\frac{\partial q}{\partial x} - \chi\frac{q^2}{h^2}\frac{\partial h}{\partial x} = -\frac{\kappa\nu}{\delta}\frac{q}{h^2} + gh + \frac{\sigma h}{\rho}\frac{\partial^3 h}{\partial x^3}, \tag{20}$$

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{21}$$

Represent the overall flow as:

$$q = q_0 + q', \quad h = h_0 + h', \tag{22}$$

where prime denotes the disturbed part of value, and substitute these expressions into [20] and [21]. Letting $q' \ll q_0$ and $h' \ll h_0$ and retaining the terms of the order of h'^2 and q'^2 and higher, we obtain the nonlinear equations for the thickness and flow rate disturbances with the nonlinear terms in the right-hand side:

$$\frac{\partial q'}{\partial t} + \frac{2\chi q_0}{h_0}\frac{\partial q'}{\partial x} - \chi\frac{q_0^2}{h_0^2}\frac{\partial h'}{\partial x} + \frac{\kappa\nu}{\delta h_0^2}q' - 3gh' - \frac{\sigma h_0}{\rho}\frac{\partial^3 h'}{\partial x^3} \tag{23}$$

$$= \frac{3gh'^2}{h_0} - \frac{2}{h_0}h'\frac{\partial q'}{\partial t} - \frac{2\chi q_0}{h_0^2}\left[h'\frac{\partial q'}{\partial x} + \frac{h_0}{q_0}q'\frac{\partial q'}{\partial x} - q'\frac{\partial h'}{\partial x}\right], \quad \frac{\partial h'}{\partial t} + \frac{\partial q'}{\partial x} = 0. \tag{24}$$

From the system of [23] and [24] turn to one nonstationary equation for thickness disturbances. For this purpose let [23] be differentiated with respect to x and the derivative $\partial q'/\partial x$ in the linear terms be replaced via continuity equation [24]:

$$\begin{aligned}\frac{\partial^2 h'}{\partial t^2} &+ \frac{2\chi q_0}{h_0}\frac{\partial^2 h'}{\partial x \partial t} + \chi\frac{q_0^2}{h_0^2}\frac{\partial^2 h'}{\partial x^2} + \frac{\kappa\nu}{\delta h_0^2}\frac{\partial h'}{\partial t} + 3g\frac{\partial h'}{\partial x}\\ &+ \frac{\sigma h_0}{\rho}\frac{\partial^4 h'}{\partial x^4} = -\frac{6g}{h_0}h'\frac{\partial h'}{\partial x} + \frac{2}{h_0}\frac{\partial}{\partial x}\left(h'\frac{\partial q'}{\partial t}\right)\\ &+ \frac{2\chi q_0}{h_0^2}\frac{\partial}{\partial x}\left[h'\frac{\partial q'}{\partial x} + \frac{h_0}{q_0}q'\frac{\partial q'}{\partial x} - q'\frac{\partial h'}{\partial x}\right].\end{aligned} \tag{25}$$

To exclude q' and $\partial q'/\partial t$ from the nonlinear terms the following considerations will be used. In continuity equation the variables x and t are substituted by ξ and t, where $\xi = x - Ct$ and C is the wave velocity assumed to be constant for quasistationary waves. The experiments show that in many cases the observed waves may be considered as weak-dispersive and weak-nonlinear. Then

$$\frac{\partial h'}{\partial t} - C\frac{\partial h'}{\partial \xi} + \frac{\partial q'}{\partial \xi} = 0. \qquad [26]$$

For the quasistationary process the wave profile in a moving coordinate system deforms only slightly. As a result we pass from [26] to the approximate equation $C\partial h'/\partial\xi = \partial q'/\partial\xi$, then

$$q' = Ch', \qquad [27]$$

$$\frac{\partial}{\partial t} = -C\frac{\partial}{\partial x}. \qquad [28]$$

For the stationary waves equations [27] and [28] are exact.

Now substitute [27] into the nonlinear terms of [25], which at Re $= 1 - 1/\epsilon^2$ always have a lower order of magnitude than the main terms. The derivatives of the form $C\partial/\partial x$ appeared in the nonlinear terms will be replaced according to [28]. As a result we obtain the following nonlinear nonstationary thickness disturbance equation:

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + C_0\frac{\partial}{\partial x}\right)h' + \frac{\delta}{\kappa}\frac{h_0^2}{\nu}\left(\frac{\partial}{\partial t} + C_1\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + C_2\frac{\partial}{\partial x}\right)h' \\ &\quad + \frac{6\delta h_0 g}{\kappa\nu}h'\frac{\partial h'}{\partial x} - 2\delta^2\frac{(\chi - 1)}{\kappa^2}\frac{gh_0^3}{\nu^2}\frac{\partial}{\partial t}\left(h'\frac{\partial h'}{\partial t}\right) \\ &\quad + \frac{\delta}{\kappa}\frac{\sigma}{\rho\nu}h_0^3\frac{\partial^4 h'}{\partial x^4} = 0,\end{aligned} \qquad [29]$$

where $C_0 = 3q_0/h_0$, $C_1 = q_0(\chi + \sqrt{\chi^2 - \chi'}/h_0$, $C_2 = q_0(\chi - \sqrt{\chi^2 - \chi'})/h_0$.

Equation [29] has a typical two-wave structure, i.e. the wave process on a liquid film includes a lower-order wave with velocity C_0 and waves described by the derivatives of higher orders with velocities C_1 and C_2.† The derivation and analysis of such equations are discussed in detail by Whitham (1974). Show that [29] may be used for the analysis of the falling film stability and the process of nonlinear wave formation.

In further considerations approximation [19] will be used for the velocity profile in a vertical falling liquid film. This profile will be utilized to calculate the coefficients:

$$\delta = 2/3, \quad \kappa = 2, \quad \chi = 1.2, \quad C_1 = 1.69u_0,$$
$$C_2 = 0.71u_0, \quad u_0 = q_0/h_0 = gh_0^2/(3\nu).$$

After substituting these coefficients into [29] we obtain:

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + 3u_0\frac{\partial}{\partial x}\right)h' + \frac{2gh_0}{\nu}h'\frac{\partial h'}{\partial x} - \frac{2}{45}\frac{h_0}{\nu}\frac{\partial}{\partial t}\left(h'\frac{\partial h'}{\partial t}\right) \\ &\quad + \frac{1}{3}\frac{h_0^2}{\nu}\left(\frac{\partial}{\partial t} + 1.69u_0\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + 0.71u_0\frac{\partial}{\partial x}\right)h' + \frac{\sigma h_0^3}{3\rho\nu}\frac{\partial^4 h'}{\partial x^4} = 0,\end{aligned} \qquad [30]$$

†The physics of the C_1 and C_2 appearance is connected with the loss of stability by the initial flow in the sense of the Orr-Sommerfeld Stability.

or in the dimensionless form:

$$\left(\frac{\partial}{\partial \bar{t}} + 3\frac{\partial}{\partial \bar{x}}\right)H + 6H\frac{\partial H}{\partial \bar{x}} - \frac{2}{15}\mathrm{Re}\left(\frac{h_0}{L}\right)\frac{\partial}{\partial \bar{t}}\left(H\frac{\partial H}{\partial \bar{t}}\right) + \frac{\mathrm{Re}}{3}\left(\frac{h_0}{L}\right)\left(\frac{\partial}{\partial \bar{t}} + 1.69\frac{\partial}{\partial \bar{x}}\right)\left(\frac{\partial}{\partial \bar{t}} + 0.71\frac{\partial}{\partial \bar{x}}\right)H + \mathrm{We}\left(\frac{h_0}{L}\right)^3\frac{\partial^4 H}{\partial \bar{x}^4} = 0. \qquad [31]$$

From [31] one can easily pass to the conventional form of writing with one scale h_0, letting $L = h_0$.

If one assumes that the wave is stationary, i.e. $h' = h'(x - Ct)$, [31] will correspond to the Shkadov's (1967) and Lee's (1969) equations.

Consider the case $h_0/L \ll 1$, $\mathrm{Re} \sim 1$. It can be seen that the wave process is based on the kinematic wave $\partial H/\partial \bar{t} + 3\partial H/\partial \bar{x} = 0$. According to Whitham (1974), substitute the time derivative in the higher-order wave using the relation $\partial/\partial \bar{t} = -3\partial/\partial \bar{x}$, neglect the second nonlinear term and thereby pass from [31] to the equation

$$\frac{\partial H}{\partial \bar{t}} + 3\frac{\partial H}{\partial \bar{x}} + 6H\frac{\partial H}{\partial \bar{x}} + \mathrm{Re}\left(\frac{h_0}{L}\right)\frac{\partial^2 H}{\partial \bar{x}^2} + \mathrm{We}\left(\frac{h_0}{L}\right)^3\frac{\partial^4 H}{\partial \bar{x}^4} = 0. \qquad [32]$$

This type equation has recently been used by Gjevik (1970), Maurin *et al.* (1977), Nepomnyaschii (1974) and Petviashvili & Tzvelodub (1978) as the basic equation for the analysis of nonlinear waves on vertical falling films.

Come back to [31] and consider the case when $\mathrm{Re} \gg 1$ and the wave process is based on the second-order waves. Separating the streamwise waves $\partial H/\partial \bar{t} + 1.69\ \partial H/\partial \bar{x} = 0$, substituting $\partial/\partial \bar{t} = -1.69\ \partial/\partial \bar{x}$ and integrating over x, we arrive at:

$$\frac{\partial H}{\partial \bar{t}} + 1.69\frac{\partial H}{\partial \bar{x}} + 2.07H\frac{\partial H}{\partial \bar{x}} - 4.01\left(\frac{L}{h_0}\right)\frac{H}{\mathrm{Re}} - \frac{9.2}{\mathrm{Re}}\left(\frac{L}{h_0}\right)H^2 - 3.06\frac{\mathrm{We}}{\mathrm{Re}}\left(\frac{h_0}{L}\right)^2\frac{\partial^3 H}{\partial \bar{x}^3} = 0. \qquad [33]$$

This equation was obtained by Nakoryakov & Shreiber (1973) as a model to describe the film surface wave at high Re.

Thus in the case of a longwave process at low Reynolds numbers the energy transfer from the mean flow to a kinematic wave follows the higher-order wave mechanism. This leads to the appearance of the energy source term with the second derivative (or "negative" viscosity term) in [32].

At high Re the energy is transferred into the higher-order wave, which can be referred to as "inertial," by the kinematic wave (a linear term of "low-frequency" energy source in [33]).

The analysis of the film surface waves in a wide range of Reynolds number should be made on the basis of [31]. The exact range of its applicability can be established only from the comparison between its solutions and the experiments and accurate numerical calculations. Such a comparison for the linear waves is given in section 6.

3. LINEAR ANALYSIS OF FILM FLOW STABILITY

Let the linear equation corresponding to [31] be written as

$$\frac{\partial H}{\partial \tilde{t}} + 3\frac{\partial H}{\partial \tilde{x}} + \frac{\mathrm{Re}}{3}\left(\frac{\partial}{\partial \tilde{t}} + 1.69\frac{\partial}{\partial \tilde{x}}\right)\left(\frac{\partial}{\partial \tilde{t}} + 0.71\frac{\partial}{\partial \tilde{x}}\right)H + \mathrm{We}\frac{\partial^4 H}{\partial \tilde{x}^4} = 0. \qquad [34]$$

From [34] derive dispersion equations for the temporally growing (damping) waves letting H as:

$$H = A_0 \exp[i(k\tilde{x} - \Omega\tilde{t})] = A_0 \exp[ik(\tilde{x} - \tilde{C}\tilde{t})] \cdot \exp \beta\tilde{t},$$

where $k = 2\pi h_0/\lambda$ is the real wave number, $\Omega = \omega + i\beta$ is the complex frequency which is made dimensionless by using h_0 and u_0, and $\tilde{C} = C/u_0$ is the real part of phase velocity. After substituting H into [34] and separating real and imaginary parts we arrive at

$$-\tilde{C} + 3 - \tfrac{2}{3}\tilde{C}\beta \mathrm{Re} + 0,8\beta\,\mathrm{Re} = 0, \qquad [35]$$

$$3\beta - k^2\,\mathrm{Re}(\tilde{C}^2 - 2,4\tilde{C} + 1,2) + \mathrm{Re}\beta^2 + 3\,\mathrm{We} \cdot k^4 = 0. \qquad [36]$$

Similar equations were derived by Shkadov (1968) directly from the system of [17] and [18], but the dispersion equations had not been analyzed.

From [35] it follows that

$$\beta\,\mathrm{Re} = -\frac{3}{2}\frac{\tilde{C} - 3}{\tilde{C} - 1,2}. \qquad [37]$$

Excluding β from [36] with the help of [37], derive the quadratic (with respect to $k^2\mathrm{Re}^2$) equation

$$(k\,\mathrm{Re})^4 - (k\,\mathrm{Re})^2\frac{\mathrm{Re}^3}{3\,\mathrm{We}} \cdot (\tilde{C} - \tilde{C}_1)(\tilde{C} - \tilde{C}_2) - \frac{3\,\mathrm{Re}^3\,(\tilde{C} - 3)(\tilde{C} + 0.6)}{4\,\mathrm{We}\,(\tilde{C} - 1.2)^2} = 0,$$

where $\tilde{C}_1 = 1.69$, $\tilde{C}_2 = 0.71$.
Its solution is:

$$(k \cdot \mathrm{Re})^2 = \frac{\mathrm{Re}^3}{6\,\mathrm{We}}(\tilde{C} - \tilde{C}_1)(\tilde{C} - \tilde{C}_2) \cdot \left[1 \pm \sqrt{1 + \frac{27\,\mathrm{We}\,(\tilde{C} - 3)(\tilde{C} + 0.6)}{\mathrm{Re}^3\,(\tilde{C} - 1.2)^2(\tilde{C} - \tilde{C}_1)^2(\tilde{C} - \tilde{C}_2)^2}}\right]. \qquad [38]$$

The results of computer calculations are given in figures 2 and 3.

From [36] and [37] it follows that the neutral waves exist if

$$\beta = 0, \quad \tilde{C} = 3, \quad k = \sqrt{\mathrm{Re}/\mathrm{We}}, \qquad [39]$$

Waves with $\tilde{C} > 3$ are exponentially damping, while those with $\tilde{C} < 3$ grow. The asymptotics of the dispersion curves in the region of $\tilde{C} > 3$ is obtained from [38], provided that the right-hand term in the radicand is neglected

$$\tilde{C} = 1.2 + 0.49\sqrt{1 + \frac{25}{2}\frac{\mathrm{We}}{\mathrm{Re}}k^2}.$$

For Re ~ I and not very low k (capillary ripples in front of the large waves propagating over the thin residual layer) we have

$$\tilde{C} = 1.2 + k\sqrt{3\,\mathrm{We}/\mathrm{Re}}. \qquad [40]$$

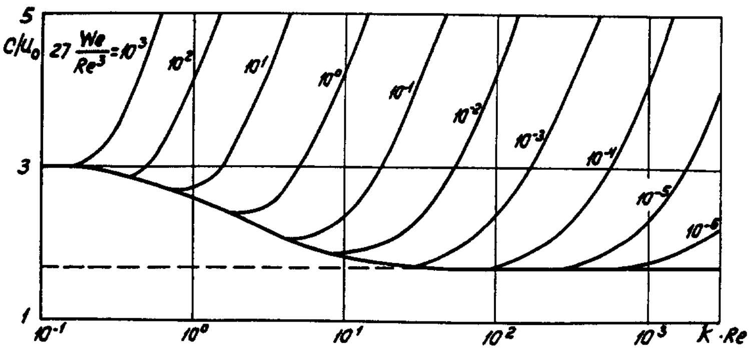

Figure 2. Dispersion curves for two-dimensional waves on vertical falling liquid film.

In the coordinate system moving at a velocity of $1.2u_0$, [40] exactly coincides with the dispersion equation for the capillary waves on "shallow" water.

The analysis of [37] shows that the maximum growth rate corresponds to the minimum phase velocity. Hence the curve of the fastest growing waves intersects the dispersion curves in figure 2 at the points of minimum phase velocities. For an accurate determination of the characteristics of fastest growing waves come back to [36] and [37].

Rewrite [37] as:

$$\tilde{C} = 1.2 + 1.8/\phi, \tag{41}$$

where $\phi = 1 + \frac{2}{3}\beta \, \mathrm{Re} \geq 1$.

After substituting [41] into [36], differentiate the obtained expression with respect to k, and allowing for the extremum condition $\partial\phi/\partial\kappa = 0$, we obtain

$$k \cdot \mathrm{Re} = 0.2\sqrt{\frac{\mathrm{Re}^3}{\mathrm{We}}\left(\frac{13.5}{\phi^2} - 1\right)} = \sqrt[4]{\frac{3}{4}\frac{\mathrm{Re}^3}{\mathrm{We}}(\phi^2 - 1)} \tag{42}$$

Finally, after substituting [41] and [42] into [36] we arrive at:

$$\frac{\mathrm{Re}^3}{\mathrm{We}} = \frac{\phi^4(\phi^2 - 1)}{(\phi^2 - 13.5)^2}\frac{3 \cdot 10^3}{6.4}. \tag{43}$$

The maximum growth rate is

$$\beta = \frac{3(\phi - 1)}{2\,\mathrm{Re}}. \tag{44}$$

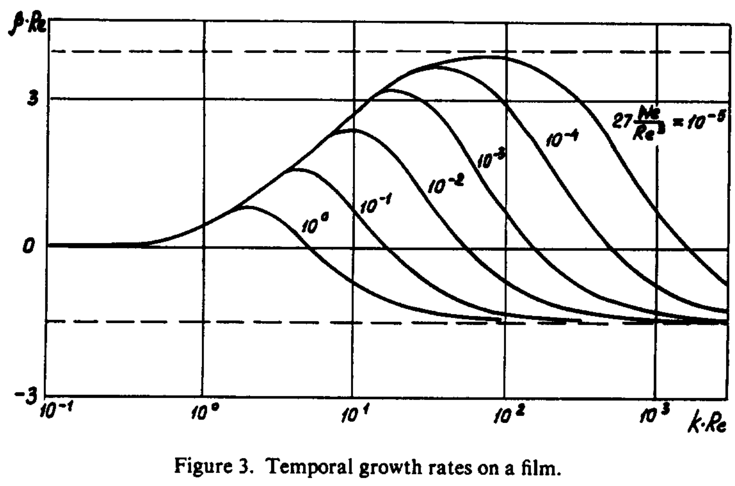

Figure 3. Temporal growth rates on a film.

To pass from the temporal growth rate β to the spatial growth rate $(-\alpha)$ measured in the experiments, the known Gaster's transformation:

$$-\alpha = \beta \Big/ \frac{\partial \omega}{\partial k} = \beta \Big/ \left[\tilde{C} + k\frac{\partial \tilde{C}}{\partial k}\right] = \beta / \tilde{C} \qquad [45]$$

should be used. Here it has been taken into account that for the fastest growing waves $\partial \tilde{C}/\partial k = 0$. The numerical calculation shows that in terms of the problem formulates the temporal and spatial growth rates are related by [45] with sufficient accuracy.

Thus the system of [41–45] describes all the characteristics of the fastest growing waves.

Since Re and We enter into the dispersion equation as Re^3/We ratio, it should be transformed so that only one flow rate parameter, Re, will be used.

$$Re^3/We = 3^{2/3}(Re/Fi^{1/11})^{11/3}.$$

4. EXPERIMENTAL PROCEDURE

Wave experiments in the inception region were performed on a set-up described by Pokusaev & Alekseenko (1977). Liquid fed out of the constant-level tank through a liquid distributor is falling over the outer surface of the test section, which is a 1 m long plexiglass tube (Ø = 60 mm). Due to the experimental requirements, for the instantaneous velocity profile measurements we used a stainless steel tube (Ø = 60.8 mm) with high-polished (mirror) surface.

Liquid was fed to the test section through a 70 mm long and 0.5–1 mm wide annular orifice. The main experimental difficulty was to ensure a two-dimensional flow of the wave liquid film. For the uniform humidification the test section was strictly vertical and the annular gap and coaxility between the liquid distributor and the test section were fine-adjusted until the two-dimensional (annular) waves were obtained. This adjustment was possible due to a small clearance between the setting surfaces of the test section and the liquid distributor.

Liquid film is extremely sensitive to the external disturbances, e.g. to vibrations induced by the operating pump. Therefore the experiments were performed only with a switched-off pump and liquid was pumped into the upper tank periodically in an automatic regime.

As a working liquid we use water-glycerine solutions since they are less affected by the surfactants adsorbing on the liquid film as compared to pure water. In addition the two-dimensional waves on a water-glycerin film are more stable to three-dimensional disturbances.

Experimentally measured were: instantaneous and mean thickness of the film, wave amplitudes, velocities and lengths and the instantaneous velocity profile in a wave flow regime.

The film thickness was measured by a shadow method (figure 4). Power light source I (mercury lamp) via condenser 2 tangentially irradiates test section 3 (as illustrated by figure 4) and liquid film 4 falling over the outer surface of the vertical tube forms a shadow. The magnified film shadow pulsations are projected by objective lens 5 to photoelectron multiplier 6 and recorded in the analog or digital form.

The phase velocity of the waves was measured by the phase shift between two simultaneous recordings of the instantaneous film thickness which correspond to two different points along the tube.

The accuracy of the absolute thickness and phase velocity measurements is 2–5 and 5–9%, respectively.

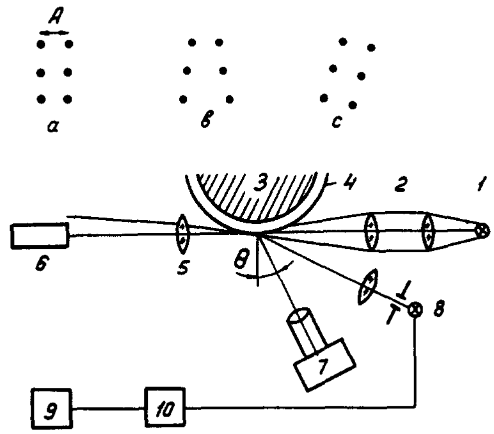

Figure 4. Scheme of film thickness and velocity profile measurements.

The instantaneous velocity field in a wave liquid film was estimated by two synchronized methods, they are the shadow method for thickness determination and stroboscopic particle visualization for velocity measurements. The latter was first used by Cook & Clark (1971) and Ganchev *et al.* (1972) only for the mean velocity profile measurements. The measurement of instantaneous velocity profile in a wave liquid film is schematically shown in figure 4. Liquid film 4 with small concentrations of 1–5 μ round aluminium particles is falling over the outer surface of stainless steel high-polished tube 3. If the particles are recorded by camera 7 at side pulse irradiation by lamp 8, the film frame fixes a discontinuous track of one particle from which, knowing the frequency and the magnification factor, the particle velocity can be defined. The pulse frequency of lamp 8 is set by sonic frequency generator 8 triggering stroboscope 10.

With a mirror surface of the test section and photographing at a θ angle to the normal to the surface (figure 4) the camera will record not only tracks of the real particle but also those of the particle virtual image, formed by the mirror, as illustrated in figures 4a–c. By simple geometric constructions the formula:

$$y = \frac{A}{2N\sin\theta}\sqrt{n^2 + \tan^2\theta\,(n^2 - 1)} \qquad [46]$$

is derived, where N is the magnification factor measured without liquid in the plane parallel to the frame, n is the liquid refractivity and A is the distance between real and virtual particle images on the photograph film.

Figure 4a–c illustrates typical double tracks when the pulse number is 3. Case a is a smooth film with only a longitudinal velocity component calculated by the formula:

$$u = f \cdot (x_{i+1} - x_i)/N, \qquad [47]$$

where x_i is the longitudinal coordinate of the particle image on the fim at the ith pulse, f is the pulse frequency. Case b is a smooth film but also with a transverse velocity component equal to

$$v = (y_{i+1} - y_i) \cdot f, \qquad [48]$$

where y_i is the transverse particle coordinate calculated by formula [46]. Figure 4 c is the wave liquid film with the surface inclination at a certain angle to the longitudinal axis x.

The analysis of the profiles of two-dimensional waves observed in the experimental with low Reynolds number, shows that this angle reaches the maximum value (23°) only in the region of the leading front of the largest waves. In this case an additional error of the y and u determination is 3 and 6.5%, respectively ($\theta = 30°$). For small and large waves (except for their leading front) formulas [46–48] can be used. The main experimental errors are: 7% for the absolute y values, 2% for its relative values and 4–5% for the longitudinal velocity.

The particle position in a wave was determined by simultaneous recording of the film thickness and the moments of lamp pulses.

As a rule the photographic film frame contains 5–10 tracks responsible for a small part of the film. The velocity field over the total length of the wave was constructed on the basis of 30–50 frames. It is very important that the waves will be strict-regular and two-dimensional, since the velocity distribution is estimated by a set of data for various waves.

5. INSTANTANEOUS VELOCITY PROFILE IN A WAVE LIQUID FILM

Numerous literature data are known for liquid velocities in a film, but the literature lacks the results of instantaneous velocity profile measurements in a wave liquid film, though these data are of special interest for the theory formulation.

As noted above, the method of velocity measurements requires regular periodic waves. For this purpose the film flow was artificially disturbed by liquid flow rate pulsations, as done by Kapitza & Kapitza (1949) and Pokusaev & Alekseenko (1977), to form strict-regular two-dimensional stationary periodic waves. The picture of stationary excited waves at a given Re is determined only by the frequency of imposed disturbances. If the frequencies of the natural stationary waves and of the imposed disturbances coincide the wave pictures are identical, i.e. the natural wave regime is a special case of the stationary excited waves.

The results of measurements of instantaneous longitudinal velocity profiles for the characteristic value of the Re = 12.4 and two characteristic types of observed waves are given in figure 5. Above each wave profile their associated sections are shown. Points correspond to the section numbers: I. Selfsimilar parabolic profile plotted by the maximum thickness and velocity values; II. Nusselt velocity profile for a smooth laminar film plotted by the residual layer thickness. A dotted line is the wave phase velocity. The other characteristics of wave regimes are listed in table 1.

The above plots show that in the region of maximum film thicknesses the velocity profile changes insignificantly, while at the minimum h values (sections 1 and 9, figure 5 b, and sections 2, figure 5 a) it undergoes great changes. In the residual layer region the flow is purely laminar and is described by the Nusselt theory (curve II in figure 5 a). The maximum liquid velocities in a wave reach values of the wave phase velocities (figure 5 a).

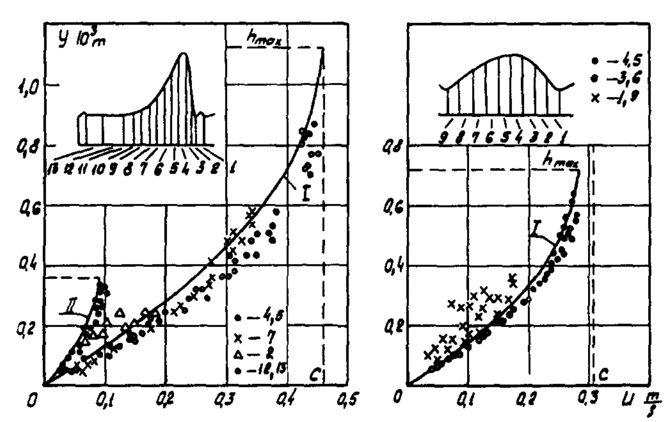

Figure 5. Instantaneous velocity profiles in a wave liquid film. Wave characteristics are listed in table 1.

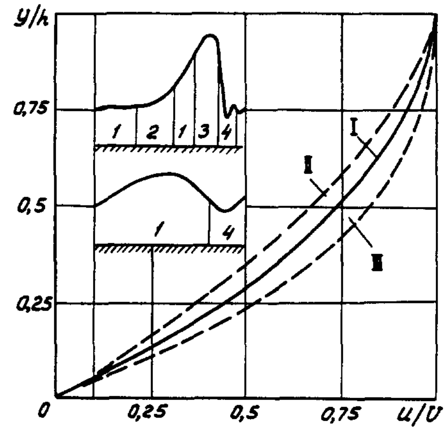

Figure 6. Dimensionless velocity profile.

Table 1.

	$\nu \cdot 10^6$ m^2/s	$\langle h \rangle$ mm	h_{max} mm	λ mm	C mm/s	$\frac{C}{q_0/\langle h \rangle}$	$\frac{\langle U \rangle}{q_0/\langle h \rangle}$
Figure 5a	7.2	0.545	1.12	36	460	2.83	1.27
Figure 5b	7.1	0.56	0.73	11.8	310	1.93	1.48

The analysis of the results shows that in a given wave section the greater longitudinal velocities correspond to the higher values of the y-coordinate. This points to the absence of a stationary vortex even for the waves with so large amplitude as shown in figure 5a (provided that $C > u_{max}$).

The results of velocity profile measurements are schematically shown in figure 6 in the dimensionless coordinates $(y/h; u/U]$ where h is the local thickness, U is the local surface velocity calculated by the velocity of particle-marks, detected near the film surface (I – selfsimilar parabolic profile). The wave profiles here are roughly divided into sections. In section I the velocity profile is described by the selfsimilar parabolic law. In section 2 the velocity profile is less filled as compared to the parabolic one, and corresponds to region II, while in section 3 it is more filled (region III). The maximum deviations from the parabolic law reach 15%. For section 4 no velocity profile has been plotted due to a great scatter of the experimental points.

6. TWO-DIMENSIONAL WAVES IN THE INCEPTION REGION

A vertical falling liquid film at Re = 5–50 can be described as follows. In close proximity to the outlet orifice the liquid film is smooth. Then at some distance from the orifice edge due to the natural instability of the smooth laminar flow, infinitesimal two-dimensional periodic disturbances arise fast growing in the amplitude. At sufficiently large amplitudes the nonlinearity is observed and the wave regime becomes stationary and nonlinear. Two-dimensional waves are unsteady and soon break into three-dimensional horseshoe disturbances which are essentially non-stationary.

Data on the evolution of two-dimensional waves in their inception region are given in figures 7–9. The film thickness oscillograms were taken at various disturbances from the outlet orifice by moving the optical system along the test section. As illustrated by figures 7 and 8, the arising waves are sinusoidal and their amplitude $a = h_{max} - h_{min}$ first grows exponentially with distance and then becomes constant. For the sake of convenience data on the velocity and wavelengths are given in figure 9 as a function of the amplitude which directly shows the linearity of waves in the inception region.

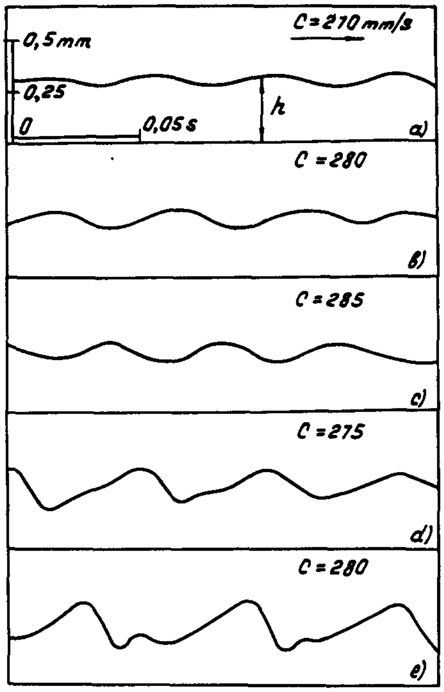

Figure 7. Wave evolution on a vertical falling liquid film: $\nu = 2.34 \cdot 10^{-6}$ m²/s, $\sigma/\rho = 60.2 \cdot 10^{-6}$ m³/s², Re = 15.3, x/h_0 = 155 (a), 165 (b), 185 (c), 200 (d), 240 (e).

It should be noted that the arising waves are not strict-regular, therefore to obtain average wave characteristics the signal should be statistically analyzed. However, with the properly organized liquid feed its suitable properties and flow rates, fairly regular two-dimensional waves can be observed in the inception region. Where possible, we considered just these regimes.

According to the linear theories of wave instability, the waves actually observed near the wave inception line, should correspond to the fastest growing waves which is partially confirmed in several publications. Thus Pierson & Whitaker (1977) and others reported on the experimental data only for the wave lengths and velocities on water films, Krantz & Goren (1971) performed measurements only at Re $\lesssim$ 1 for oil films. Figures 10–12 generalize our and other authors' experimental results on the growth rates, velocities and lengths of the growing waves and compare them with the linear theories of the fastest

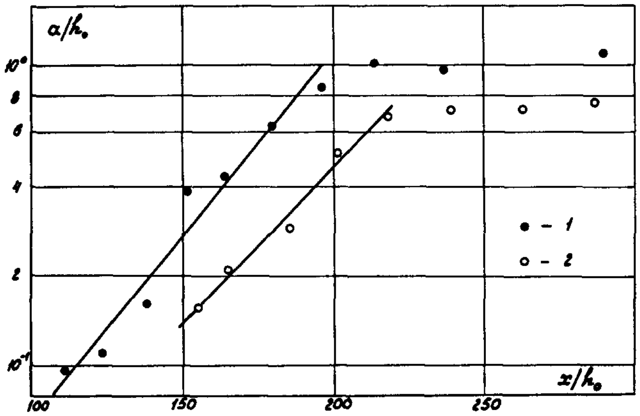

Figure 8. Amplitude of growing waves: $\nu = 2.34 \cdot 10^{-6}$ m²/s, $\sigma/\rho = 60.2 \cdot 10^{-6}$ m³/s²; Re = 36.4 (I), 15.3 (2).

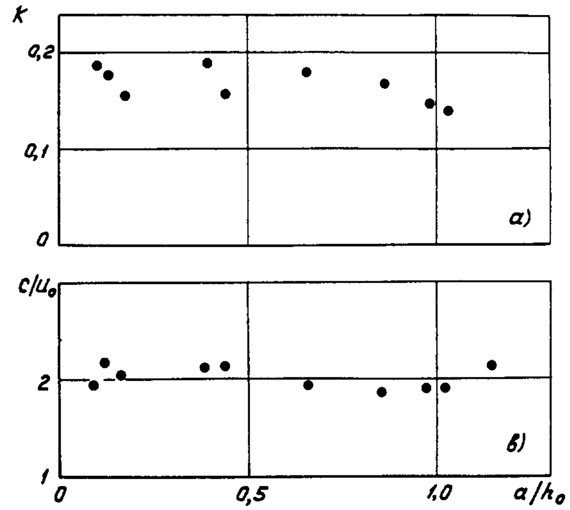

Figure 9. Wave number (a) and phase velocity (b) of growing waves: $\nu = 2.34 \cdot 10^{-6}$ m^2/s, $\sigma/\rho = 60.2 \cdot 10^{-6}$ m^3/s^2.

growing waves. For plotting the coordinates were chosen so that our theoretical dependences be universal curves.

The growth rate was estimated by the tangent of the inclination angle of the lines in figure 8 plotted in the semilog coordinates:

$$-\alpha = \left[\ln \frac{\alpha_1 - \alpha_2}{h_0}\right] \Big/ \left(\frac{x_1 - x_2}{h_0}\right).$$

The approximating lines were obtained by the least squares method.

In the region of Re > 10 our experimental data for a water-glycerin solution are in agreement with Portalski & Clegg (1972). At Re $\lesssim$ 1 the Krantz & Goren's data (1971) for oil films are given.

At Re/Fi$^{1/11}$ < 0.5 the experimental points are well described by various theories: I. Present study; III. Benjamin's theory (1957); IV. Benjamin's longwave approximation (1957); and VI. Pierson & Whitaker (1977). It should be noted that in the original Benjamin's study (1957) the growth rate in formula (5.10) is erroneous (later he corrected the mistake), i.e. 0.224 should be substituted by 0.448. In the range of moderate Reynolds numbers at Re/Fi$^{1/11}$ > 2 the experimental data are well generalized by theoretical relations I and VI and partially by II.

In figures 11 and 12 neutral curves are also presented for the wave number and velocity which, however, significantly deviate from the experimental points. The experimental velocities are greatly scattered due to the difficulty of wave characteristics measurements for small-amplitude waves of very smooth slope form.

Table 2.

No.	Authors	Fluid	$\nu \cdot 10^6$ m^2/sec	$\sigma/\rho \cdot 10^6$ m^3/sec^2	Fi$^{1/11}$	Re
1	Present study	water–glycerin solution	2.12	65.3	6.78	10–40
2	Present study	water–ethanol solution	2.12	28.5	5.42	10–27
3	Present study	water–glycerin solution	3.72	61	5.46	8–48
4	Present study	water solution of ethanol and glycerin	2.34	60.2	6.4	15–36
5	Jones & Whitaker (1966)	water			9.54	6–70
6	Strobel & Whitaker (1969)	water			9.54	6–70
7	Krantz & Goren (1971)	mineral oil			1 72	0.5–5.5
8	Krantz & Goren (1971)	mineral oil			1.14	0.25–1.2

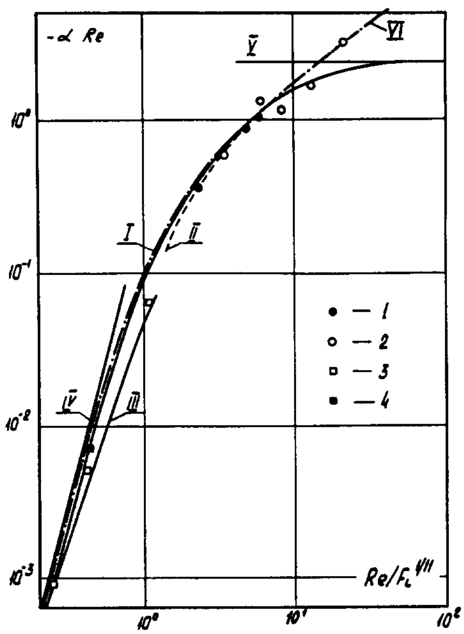

Figure 10. Spatial growth rates. Experiment: 1. $\nu = 2.34 \cdot 10^{-6}$ m^2/s, $\sigma/\rho = 60.2 \cdot 10^{-6}$ m^3/s^2; 2. Portalski & Clegg (1972), $Fi^{1/11} \approx 5.33$; 3. Krantz & Goren (1971), oil $Fi^{1/11} \approx 1.19$; 4. Krantz & Goren (1971), oil $Fi^{1/11} \approx 1.72$. Theory: I. present study (43)–(45); II. Whitaker (1964), water; III. Benjamin (1957); IV. longwave approximation, Benjamin (1957); V. Nakoryakov & Shreiber (1973) (linearized equation [33]); VI. Pierson & Whitaker (1977).

The plots given in figures 7–12 show that in the wave inception region the behaviour of growing waves at the initial stage of their evolution is described by the linear theories of the fastest growing waves. Curves I in figures 10–12, despite the simplicity of the equations used for their derivation, fairly well generalize the experimental points and are in agreement with the other theories in a wide range of $Re/Fi^{1/11}$ variations, which is one of the arguments in favour of two-wave equation [31].

Letting the results of numerical calculation of the Orr–Sommerfeld equation in the Pierson & Whitaker study (1977) be valid in a given range of Re, from the comparison of curves I and VI in figure 10 a more accurate conclusion on the applicability field of the boundary layer approximation for a liquid film can be made. As seen, the best agreement

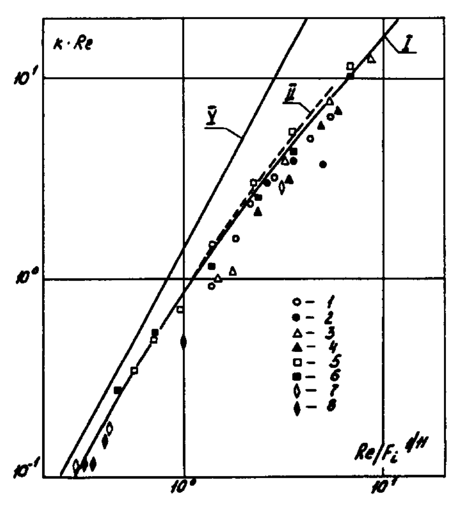

Figure 11. Wave number of growing waves. Experiment: notations 1–8 in table 2. Theories of fastest growing waves: I. present study, [42]–[43]. II. Whitaker (1964), water; neutral curve V. present study [39].

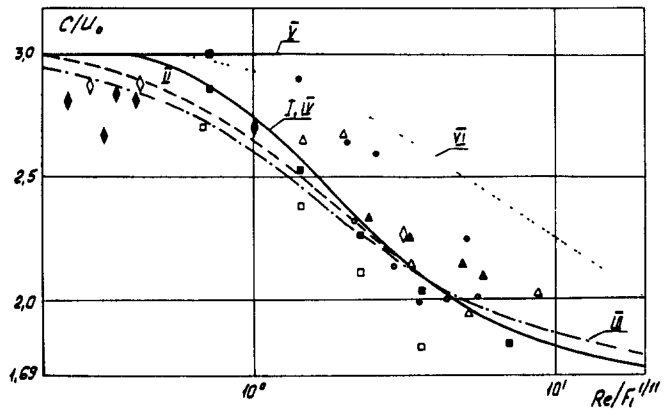

Figure 12. Phase velocity of growing waves. Notations of experimental points—figure 11 and table 2. Theories of fastest growing waves: I. present study, [41]–[43]; II. Whitaker (1964), water; III. Krantz & Goren (1971), oil, $Fi^{I/II} = 1.72$; IV. Krantz & Goren (1971), water. Neutral curves: I. present study [39] and longwave approximation Benjamin (1957); VI. Krylov et al. (1969), water.

between I and VI is observed in the range of $Re/Fi^{I/II} = I - 10$. However at $Re/Fi^{I/II} < I$ the correlation between the theories can also be considered as fairly reasinable, since the dependences differ only in the constant numerical coefficient but have the same asymptotics with respect to the Reynolds number. A more significant discrepancy is observed at $Re/Fi^{I/II} > 10$, since the theoretical dependences have different asymptotics. Anshus (1972) considered the asymptotic solutions of the Orr–Sommerfeld equation at $Re \to \infty$, and they are in agreement with the calculations made by Pierson & Whitaker (1977). For the comparison we present the asymptotic values of the characteristics of the fastest growing waves for two extreme cases of low and high Reynolds numbers at Fi = const.

	$Re \to 0$		$Re \to \infty$	
	Anshus	Present study	Anshus	Present study
$-\alpha \cdot Re$	const $\cdot Re^{11/3}$	const $\cdot Re^{11/3}$	const $\cdot Re^{2/3}$	const
$k \cdot Re$	const $\cdot Re^{11/6}$	const $\cdot Re^{11/6}$	const $\cdot Re^{4/3}$	const $\cdot Re^{11/12}$
c/u_0	const = 3	const = 3	const = 1.5	const = 1.69

7. CONCLUSIONS AND SIGNIFICANCE

A universal model equation to describe nonlinear nonstationary waves on the surface of liquid films in the range of Reynolds numbers $Re = 1 - 1/\epsilon^2$ (ϵ is the longwave process parameter) was derived by the method of integral relations with the application of selfsimilar velocity profiles. The equation is of the two-wave structure which implies that at low $Re \sim 1$ the energy is transferred to kinematic waves through the higher-order wave mechanism, and at $Re \approx 1/\epsilon^2 \gg 1$ dominating are the higher-order waves growing due to the kinematic ones. In the limiting cases of low and high Re and in the special case of stationary waves the above two-wave equation transforms to the known equations, Gjevik (1970), Nakoryakov & Shreiber (1973) and Shkadov (1967).

In terms of the derived equation a linear analysis of the stability was carried out to obtain the analytical expressions to describe neutral disturbances, fastest growing waves and capillary ripples observed in front of the large solitary waves.

An experimental system was developed to measure the instantaneous velocity profiles in a wave liquid film and the wave characteristics in the region of their formation. The assumed validity of the selfsimilarity of instantaneous velocity profiles was supported by the data on the instantaneous velocity field in a film for two-dimensional moderate-amplitude waves (figure 6).

Wave characteristics in the region of wave formation were generalized in the universal coordinates obtained from the analysis of the above two-wave equation (figures 10–12). The behaviour of linear growing waves on the film surface is shown to be described by the linear theories of the fastest growing waves. Calculations by these equations are in good agreement with the other known theories in a wide range of the $\mathrm{Re}/\mathrm{Fi}^{1/11}$ values except the very high Re.

The results indicate that the solutions of the full two-wave equations can describe all two-dimensional nonlinear wave regimes observed on the surface of falling liquid films, Nakoryakov *et al.* (1976).

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NOMENCLATURE

A	particle tracks spacing
A_0	wave amplitude
a	$h_{max} - h_{min}$
C	phase velocity
C_0	$3\,u_0$
C_1	$1.69\,u_0$
C_2	$0.71\,u_0$
f	frequency
$f(\eta)$	function in velocity profile expression
Fi	film number, $\sigma^3/\rho^3 g\nu^4$
g	gravitational acceleration
H	h'/h_0
h	film thickness
h_0	smooth laminar film thickness
k	wave number, $2\pi h_0/\lambda$
L	characteristic longitudinal scale
N	magnification
n	refractivity
P	pressure
q	instantaneous volume liquid flow rate per unit width of film
q_0	mean flow rate
Re	Reynolds number, q_0/ν
t	time
U	longitudinal component of surface velocity
v, u	transverse and longitudinal velcocity components
We	Weber number, $\sigma/\rho g h_0^2$
x, y	longitudinal and transverse coordinates

Greek Letters

α	spatial growth rate factor of amplitude
β	temporal growth rate
γ, δ, κ and χ	coefficients in equation [20]
ϵ	long-wave process parameter, h_0/λ
ϵ_1	amplitude disturbance parameter
η	y/h
θ	angular coordinate

λ wavelength
ν kinematic viscosity
ξ $x - Ct$
ρ liquid density
σ liquid surface tension
ϕ function in equation [41]
Ω dimensionless complex frequency, $\Omega = \omega + i\beta$
ω real part Ω

Subscripts
max maximum value
min minimum value

Superscripts
$'$ disturbed part of value
$-$ value made dimensionless by using L and u_0
$\sim$ value made dimensionless by using h_0 and u_0

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